

Spin Foam Models of Yang-Mills Theory Coupled to Gravity

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Abstract

We construct a spin foam model of Yang-Mills theory coupled to gravity by using a discretized path integral of the BF theory with polynomial interactions and the Barret-Crane ansatz. In the Euclidian gravity case we obtain a vertex amplitude which is determined by a vertex operator acting on a simple spin network function. The Euclidian gravity results can be straightforwardly extended to the Lorentzian case, so that we propose a Lorentzian spin foam model of Yang-Mills theory coupled to gravity.

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1 Introduction

Spin foam (SF) models of quantum gravity represent a promising approach for formulating a consistent theory of quantum gravity [1, 2]. It is a discretized spacetime path-integral approach, and the main result is the finiteness of the 3-geometry to 3-geometry transition amplitude for a non-degenerate space-time triangulation [3]. Incorporating matter in the SF framework has been started in [4]. However, the constructions given there were algebraic, without a direct connection to the discrete path-integral considerations. The difficulty in the path-integral approach is that the usual matter fields couple to gravity via the vierbein one-forms, while in the SF framework the fundamental field is a 2-form B which can be considered as an exterior product of two vierbein one-forms. Still, in some cases the coupling of matter can be expressed conveniently in terms of the B fields, like in the Yang-Mills (YM) theory case [4]

$$S_{YM} \propto \int \epsilon_{abcd} B^{ab} \wedge B^{cd} \text{Tr}(F^{kl} F_{kl}) \quad , \quad (1)$$

where $F_{kl} = e_k^\mu e_l^\nu F_{\mu\nu}$ and e_k^μ is the inverse of the vierbein matrix e_μ^a . F is the YM two-form field strength, and Tr is the gauge group trace.

In such cases one can use the discretized path-integral techniques developed for the $SU(2)$ BF theory by Freidel and Krasnov [5], in order to derive the SF amplitude. Note that Oriti and Pfeiffer have proposed recently a YM euclidian SF amplitude without using the path-integral approach [6]. In order to better understand their result, we will study the discretized path-integral for a general BF theory with polynomial in B interactions by using the Friedel-Krasnov approach. The algebraic structure of the corresponding SF amplitudes is such that it is straightforward to adapt them via the Barrett-Crane ansatz [7, 8] to the case when the B field is a simple bivector, i.e. when it describes a realistic spacetime geometry.

In section two we generalize the Friedel-Krasnov results to the case of an arbitrary Lie group G . In section three we perform the Barrett-Crane reduction of the results from the section two. In section four we construct the YM spin foam amplitudes. In section five we present our conclusions.

2 Path-integral approach

Consider a BF theory with a polynomial interaction, depending only on the B field

$$S = \int_M \langle B \wedge F \rangle + \int_M P(B) d^4x \quad , \quad (2)$$

where M is a four-dimensional manifold and B is a two-form taking values in the Lie algebra \mathfrak{g} of the Lie group G . $F = (dA_I + f_I^{JK} A_J \wedge A_K) T^I$ is the curvature two-form taking values in \mathfrak{g} and \langle, \rangle is an invariant quadratic form on \mathfrak{g} .

We would like to define the path integral

$$Z = \int \mathcal{D}A \mathcal{D}B e^{iS} \quad , \quad (3)$$

via the discretization procedure provided by a simplicial decomposition of M . We will use the generating functional technique, so that we introduce

$$Z[J] = \int \mathcal{D}A \mathcal{D}B e^{iS + i \int_M \langle B \wedge J \rangle} \quad (4)$$

and

$$Z_0[J] = \int \mathcal{D}A \mathcal{D}B e^{i \int_M \langle B \wedge (F + J) \rangle} \quad . \quad (5)$$

Hence

$$Z[J] = \exp \left(i \int_M P \left(-i \frac{\delta}{\delta J} \right) \right) Z_0[J] \quad , \quad (6)$$

and $Z = Z[0]$.

In order to define Z_0 let \mathcal{C} be a simplicial complex associated to M , and let \mathcal{C}^* be the dual complex. We denote by σ , τ and Δ a 4-simplex, a tetrahedron and a triangle of \mathcal{C} respectively, and by v , l and f the corresponding dual cells, i.e. a vertex, a link and a face, so that

$$v = \sigma^* \quad , \quad l = \tau^* \quad , \quad f = \Delta^* \quad . \quad (7)$$

Let B field be piece-wise constant in the 4-polytopes formed by the pairs (Δ, Δ^*) such that the only non-zero components of B lay in the Δ planes[†]. Then

$$S = \sum_{\Delta} \langle B_{\Delta}, F_f \rangle + \sum_{\Delta, \dots, \Delta'} C_{I \dots I'}(\Delta, \dots, \Delta') B_{\Delta}^I \cdots B_{\Delta'}^{I'} \quad , \quad (8)$$

[†]This discretization procedure is more suitable for treating interactions than the one used in [5], which was based on the delta-function B fields. See [9] for basic notions about the forms on simplicial complexes.

where

$$B_\Delta = \int_\Delta B \quad , \quad F_f = \int_f F \quad . \quad (9)$$

Given the variables (9), we define

$$Z_0[J] = \int \prod_l dA_l \prod_\Delta dB_\Delta e^{i \sum_\Delta \langle B_\Delta, F_f + J_f \rangle} \quad , \quad (10)$$

where $A_l = \int_l A$. Following the ref. [5] we will fix the measures of integration dA and dB such that

$$Z_0[J] = \int \prod_l dg_l \prod_f \delta \left(\prod_{l \in f} \left(g_l e^{J_f^{(v)}} \right) \right) \quad , \quad (11)$$

where $g_l = e^{A_l}$, and $J_f^{(v)}$ are the Lie algebra elements associated to the vertices v of the face f such that

$$\sum_{v \in f} J_f^{(v)} = J_f \quad . \quad (12)$$

The formula (11) differs from the one given in [5] by the absence of the factor $\prod_{v \in f} P(J_f^{(v)})$. We do not put this factor because the simpler expression (11) is also invariant under the gauge transformations

$$g_l \rightarrow h_v g_l h_{v'}^{-1} \quad , \quad J_f(v) \rightarrow h_v J_f(v) h_v^{-1} \quad , \quad h_v, h_{v'} \in G \quad , \quad (13)$$

where v and v' are the ends of the link l [5]. Also when $J \rightarrow 0$ the expression (11) gives the standard formula for the partition function for the BF theory [10, 1]. Hence we take the expression (11) as the definition of $Z_0(J)$.

The expression (11) can be further simplified by taking

$$J_f^{(v)} = J_f^{(v')} = \dots = \frac{J_f}{n_f} \quad , \quad (14)$$

where n_f is the number of vertices of the face f . By rescaling $J_f \rightarrow n_f J_f$ we can get rid of the n_f factors. By using the group theory formulas

$$\delta(g) = \sum_\Lambda \dim \Lambda \chi_\Lambda(g) \quad , \quad (15)$$

and

$$\int_G dg D_{\alpha_1 \beta_1}^{(\Lambda_1)}(g) D_{\alpha_2 \beta_2}^{(\Lambda_2)}(g) D_{\alpha_3 \beta_3}^{(\Lambda_3)}(g) D_{\alpha_4 \beta_4}^{(\Lambda_4)}(g) = \sum_\iota C_{\alpha_1 \alpha_2 \alpha_3 \alpha_4}^{\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4(\iota)} \left(C_{\beta_1 \beta_2 \beta_3 \beta_4}^{\Lambda_1 \Lambda_2 \Lambda_3 \Lambda_4(\iota)} \right)^* \quad , \quad (16)$$

where $D^{(\Lambda)}$ is the group representation matrix in a representation Λ , and χ_Λ is the corresponding trace, as well as the associated graphical calculus [5, 1], one arrives at

$$Z_0 = \sum_{\Lambda_f, \iota_l} \prod_f \dim \Lambda_f \prod_l C(\iota_l) \prod_v \Psi_5(\Lambda_{f(v)}, \iota_{l(v)}, e^{J_{f(v)}}) \quad , \quad (17)$$

where Ψ_5 is the spin network function associated to the pentagram graph and $C(\iota)$ is a constant coming out from the recoupling theory calculus.

The problem with (17) is that the sum over the irreps diverges. This sum can be regularized in the $J = 0$ case by passing to the quantum group $G_q = U_q(\mathfrak{g})$, q root of unity [11]. One could generalize this procedure for the non-zero J case, but since in this paper we are not interesting in the topological gravity case, we will not elaborate upon this.

3 The Barrett-Crane ansatz

General relativity can be understood as a BF theory with constraints since

$$\int \sqrt{|g|} R d^4x = \int \epsilon^{abcd} e_a \wedge e_b \wedge R_{cd} = \int B^{ab} \wedge R_{ab} \quad , \quad (18)$$

where e_a are the vierbein one-forms and R_{ab} is the spin-connection curvature two-form. Hence $G = SO(4)$ in the Euclidian case or $G = SO(3, 1)$ in the Lorentzian case and the constraint is $B^{ab} = *(e^a \wedge e^b)$. In the SF framework this constraint is implemented as a restriction on the set of irreps $\{\Lambda_f\}$ such that

$$\langle T(\Lambda_f), *T(\Lambda_f) \rangle = \epsilon^{abcd} T_{ab}(\Lambda_f) T_{cd}(\Lambda_f) = 0 \quad , \quad (19)$$

where $T(\Lambda)$ are the generators of G in the representation Λ [7, 15, 8]. The corresponding solutions are called simple irreps, which we denote as N . Equivalently, N is the irrep containing an $SO(3)$ invariant vector [13].

Given the set of simple irreps $\{N_f\}$ one postulates that Z is of the same form as in the topological case

$$Z = \sum_{N_f} \prod_f A_2(N_f) \prod_l A_1(N_{f(l)}) \prod_v A_0(N_{f(v)}) \quad , \quad (20)$$

but now the amplitudes A_i are different. In the Euclidian case $N = (j, j)$, $j \in \frac{1}{2}\mathbf{Z}$, and the face amplitude $A_2 = \dim N = (2j+1)^2$ [7]. In the Lorentzian

case $A_2 = p^2 dp$, where $N = (0, p)$, $p \geq 0$ [8]. The vertex amplitude A_0 , which is associated to the pentagram spin net, is given now as

$$A_0(N_1, \dots, N_{10}) = \int_{G/H} \prod_{v=1}^5 dx_v \prod_{l=1}^{10} K_{N_l}(x_{v(l)}, x_{v'(l)}) \quad , \quad (21)$$

where x_v are the points in the homogeneous space G/H and $H = SO(3)$ [7, 12, 13, 8]. The propagators K are given as

$$K_N(x, y) = \langle 0 | D^{(N)}(g_x g_y^{-1}) | 0 \rangle \quad , \quad (22)$$

where $|0\rangle$ is the invariant vector from N , which corresponds to the identity irrep of the subgroup H contained in N . There is a freedom in choosing the edge amplitude A_1 , and a choice which makes the sum (20) finite is the spin net amplitude for the theta-four graph [14, 3].

Given these results and the formula (17) for the topological case, it is natural to propose the following expression for $Z_0(J)$ in the non-topological case

$$Z_0(J) = \sum_{N_f} \prod_f A_2(N_f) \prod_l A_1(N_{f(l)}) \prod_v \Phi_5(N_{f(v)}; e^{J_{f(v)}}) \quad , \quad (23)$$

where

$$\Phi_5(N_1, \dots, N_{10}; e^{J_1}, \dots, e^{J_{10}}) = \int_{G/H} \prod_{v=1}^5 dx_v \prod_{l=1}^{10} K_{N_l}(x_{v(l)}, x_{v'(l)}; e^{J_l}) \quad , \quad (24)$$

and the propagator

$$K_N(x, y; e^J) = \langle 0 | D^{(N)}(g_x e^J g_y^{-1}) | 0 \rangle \quad , \quad (25)$$

is the "source" propagator introduced in [13]. Hence Φ_5 is an example of the simple spin network functions introduced in [13].

4 The Yang-Mills model

The YM action on \mathcal{C} is given by [16]

$$\lambda^2 S_{YM} = \sum_{\Delta} \frac{A(\Delta^*)}{A(\Delta)} \left(\text{Re Tr } \tilde{U}(\Delta) - n \right) \quad , \quad (26)$$

where \tilde{U} is the triangle holonomy associated to the gauge group \tilde{G} and n is the dimension of the matrix \tilde{U} . $A(\Delta)$ is the area of the triangle Δ , while $A(\Delta^*)$ is the area of its dual face. Let us rewrite this action as

$$\lambda^2 S_{YM} = \sum_{\Delta} \frac{6V(\Delta, \Delta^*)}{A^2(\Delta)} \left(\text{Re Tr } \tilde{U}(\Delta) - n \right) \quad , \quad (27)$$

where $V(\Delta, \Delta^*) = \frac{1}{6}A(\Delta)A(\Delta^*)$ is the 4-volume of the 4-polytope (Δ, Δ^*) [9]. This can be further rewritten as

$$\lambda^2 S_{YM} = \frac{1}{3} \sum_{\sigma} \sum_{\Delta \in \sigma} \frac{6V(\Delta, \Delta^*)}{A^2(\Delta)} \left(\text{Re Tr } \tilde{U}(\Delta) - n \right) \quad , \quad (28)$$

which is suitable for the spin foam formalism.

The Oriti-Pfeiffer proposal [6] can be understood as an approximation

$$V(\Delta, \Delta^*) \approx \sum_{\Delta' \in \sigma} C(\Delta, \Delta') \langle B_{\Delta}, *B_{\Delta'} \rangle \quad . \quad (29)$$

This is an approximation because $V(\Delta, \Delta^*)$ depends in general on the B fields from the adjoint 4-simplices which share the polytope (Δ, Δ^*) , so that for an exact formula one would have to include the terms with $\Delta' \in \sigma'$. A further approximation is to take the C 's to be constant up to the orientation sign factors $\text{sign}(\Delta, \Delta')$. This happens for the symmetric lattices when $V(\Delta, \Delta^*) = (2/5)V(\sigma)$, where $V(\sigma)$ is the 4-volume of the 4-simplex σ . Since

$$V(\sigma) = \frac{1}{30 \cdot 4!} \sum_{\Delta, \Delta'} \text{sign}(\Delta, \Delta') \langle B_{\Delta}, *B_{\Delta'} \rangle \quad , \quad (30)$$

one obtains

$$C(\Delta, \Delta') \approx \frac{4}{30 \cdot 5!} \text{sign}(\Delta, \Delta') \quad . \quad (31)$$

One can then define

$$Z_{YM} = \int \prod_{\epsilon} d\tilde{g}_{\epsilon} \prod_l dg_l \prod_{\Delta} dB_{\Delta} e^{iS_0 + iS_{YM}} = \int \prod_{\epsilon} d\tilde{g}_{\epsilon} \tilde{Z}_{YM} \quad , \quad (32)$$

where ϵ 's are the edges of the complex \mathcal{C} . By using the previous results we obtain

$$\tilde{Z}_{YM} \approx \sum_{N_f} \prod_f A_2(N_f) \prod_l A_1(N_{f(l)}) \prod_v \left[e^{-i\beta \hat{S}_v} \Phi_5(N_{f(v)}; e^{J_{f(v)}}) \right]_{J=0} \quad , \quad (33)$$

where β is a numerical constant and

$$\hat{S}_v = \sum_{\Delta, \Delta' \in \sigma} \frac{(Re Tr \tilde{U}(\Delta) - n)}{A^2(\Delta)} \text{sign}(\Delta, \Delta') \left\langle \frac{\partial}{\partial J_f}, * \frac{\partial}{\partial J_{f'}} \right\rangle . \quad (34)$$

A natural further simplification is to take

$$A(\Delta) = \sqrt{j(j+1)} \quad , \quad (35)$$

in the Euclidian case [6]. In the Lorentzian case the labels of the triangles can be related to the areas of space-like triangles [8], and hence it is natural to take

$$A(\Delta) = \sqrt{p^2 + 1} \quad . \quad (36)$$

One can show that

$$\left[e^{-i\beta \hat{S}_v} \Phi_5(N_{f(v)}; e^{J_{f(v)}}) \right]_{J=0} = \sum_{\alpha, \beta} C_{\alpha_1 \dots \alpha_{10}}^{\beta_1 \dots \beta_{10}} \langle \alpha_1 \dots \alpha_{10} | e^{-i\beta \hat{S}_v(T)} | \beta_1 \dots \beta_{10} \rangle \quad , \quad (37)$$

where $\hat{S}(T)$ is the operator obtained from (34) by replacing the ∂J 's with the $T(N)$'s. It acts in the space $H_v = \bigotimes_{\Delta \in \sigma} V(N_f)$. The coefficients C are given as products of five Barrett-Crane intertwiners

$$S_{\alpha_1 \dots \alpha_4}^{N_1 \dots N_4} = \sum_N C_{\alpha_1 \dots \alpha_4}^{N_1 \dots N_4(N)} \quad . \quad (38)$$

From (37) it is clear that in the Euclidian case the expression $e^{-i\beta \hat{S}_v} \Phi_5|_{J=0}$ is well defined since N 's are finite-dimensional and hence all the sums are finite. The expectation value in (37) is the matrix element of an exponential of a finite matrix, which exists.

In the Lorentzian case, the operator representation (37) is not that useful for showing that $e^{-i\beta \hat{S}_v} \Phi_5|_{J=0}$ is well-defined. We expect it to be well defined because in the Euclidian case the propagators[‡]

$$K_j(x, y) = \frac{\sin(2j+1)d(x, y)}{(2j+1) \sin d(x, y)} \quad , \quad \cos d(x, y) = x \cdot y \quad , \quad (39)$$

get deformed when the source J is turned on as

$$\cos d \rightarrow \cos d \cos |J| + \sin_J d \sin |J| \quad , \quad (40)$$

[‡]The extra factor $(2j+1)^{-1}$ follows from the formula (22), since the $SU(2)$ invariant vector has a normalisation factor $(2j+1)^{-1/2}$.

where

$$\sin_J d = \cos \theta \sin \phi \cos \phi_J + \sin \theta \cos \phi \cos \theta_J \quad (41)$$

for $x = (\cos \theta, \sin \theta, 0, 0)$ and $y = (\cos \phi, \sin \phi \cos \alpha, \sin \phi \sin \alpha, 0)$. The angles θ_J and ϕ_J are determined by the 3-vectors x and J and y and J , respectively, which are associated to the corresponding $SU(2)$ subgroup elements. The deformation (40) is qualitatively similar to the deformation $d \rightarrow d + |J|$, so that it is no surprise that the integral (24) is finite, given that the $J = 0$ integral is finite. Since the Lorentzian propagator

$$K_p(x, y) = \frac{\sin p d(x, y)}{p \sinh d(x, y)} \quad , \quad \cosh d(x, y) = x \cdot y \quad , \quad (42)$$

can be obtained from the Euclidian one via analytical continuation $2j + 1 \rightarrow ip$ and $d \rightarrow id$, we then expect to have a deformation

$$\cosh d \rightarrow \cosh d \cos |J| + \sinh_J d \sin |J| \quad , \quad (43)$$

where

$$\sinh_J d = \cosh \theta \sinh \phi \cosh \phi_J + \sinh \theta \cosh \phi \cosh \theta_J \quad (44)$$

for $x = (\cosh \theta, \sinh \theta, 0, 0)$ and $y = (\cosh \phi, \sinh \phi \cos \alpha, \sinh \phi \sin \alpha, 0)$. This is similar to the deformation $d \rightarrow d + i|J|$. Since the $J = 0$ integral (24) converges, we then expect that the $J \neq 0$ integral will converge as well.

5 Conclusions

The discretized path-integral approach to spin foam amplitudes based on the generating functional technique is very useful, because the algebraic structure of the expressions one obtains in the topological case is such that it can be straightforwardly extended to the non-topological cases. This algebraic structure is the tensor category of irreducible representations of the symmetry group, so that the spin foam transition amplitudes can be understood as functors constructed from the special morphisms, which are the spin networks [17]. Even in the non-topological case, when the simple irreps N do not form a tensor category, one is actually dealing with the tensor category of the irreps of the subgroup H [17], which in the YM case are just the trivial identity irreps, so that it is natural that we have proposed the generating functional in terms of the simple spin network functions.

Because of this, we consider our proposal for the vertex amplitude (37) more appropriate for the spin foam formalism than the one proposed in [6], which is

$$\frac{\text{Tr}_{H_v}(e^{-i\beta\hat{S}_v(T)})}{\dim H_v} = \sum_{\alpha} (\dim H_v)^{-1} \langle \alpha_1 \cdots \alpha_{10} | e^{-i\beta\hat{S}_v(T)} | \alpha_1 \cdots \alpha_{10} \rangle \quad . \quad (45)$$

Although this is a reasonable proposal, this expression does not have a simple interpretation in terms of spin network evaluations. Consequently, it is not clear what would be a Lorentzian extension of this expression. On the other hand, our expression (37) is based on a simple spin network evaluation, and hence it makes sense in the Lorentzian case.

We expect that the YM amplitude (33) is finite both in the Euclidian and in the Lorentzian case because of the very rapid convergence properties of the $J = 0$ sums in the Euclidian case [18].

Note that the structure of the vertex amplitude is such that one can use the perturbative expansion for small β

$$\left[e^{-i\beta\hat{S}_v} \Phi_5 \right]_{J=0} = \left[(1 - i\beta\hat{S}_v + \cdots) \Phi_5 \right]_{J=0} = A_0 - i\beta \left[\hat{S}_v \Phi_5 \right]_{J=0} + \cdots \quad . \quad (46)$$

The second term in (46) can be interpreted as the pentagram spin network amplitude with the insertions of two endomorphisms $T(N)$ and $T(N')$ at the edges N and N' . If we neglect the higher-order terms in the expansion (46) we will obtain a spin foam amplitude which is a sum of amplitudes where a string of pentagram spin networks is connected by the matter edges carrying the adjoint representation of G . This amplitude is of the general type proposed in [4], but it is more complicated than the ones considered there. In [4], only the amplitudes where the matter edges link the vertices of the pentagrams were considered, while the amplitude corresponding to (46) has the matter edges linking the edges of the pentagrams and there are $T(N)$ insertions. This demonstrates that in order to understand completely the coupling of matter in the spin foam formalism, it is necessary to study the corresponding discretized path integral.

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